Basic Group Theory

Def Def 1 Group.

set $\{G : a, b, \ldots\}$ and $\cdot$ form a group iff

(a) $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ associative

(b) $\exists\ e \in G \ni a \cdot e = a$ for all $a \in G$

(c) For each $a \in G\ \exists\ a^{-1} \in G \ni a \cdot a^{-1} = e$

Examples: C_2 $\{e, a, a^2 = e\}$ Cyclic groups

C_3: $\{e, a, a^2, a^3 = e\}$

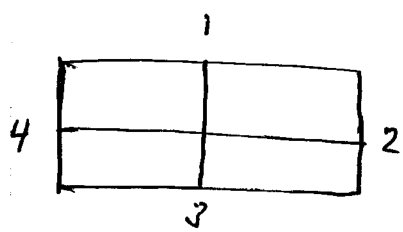
Def. Abelian $\Longleftrightarrow$ commutative i.e. $a \cdot b = b \cdot a$ $\forall a, b \in G$

Def order = number of elements.

Simplest non-cyclic group is of order 4

D_2 Dihedral Group

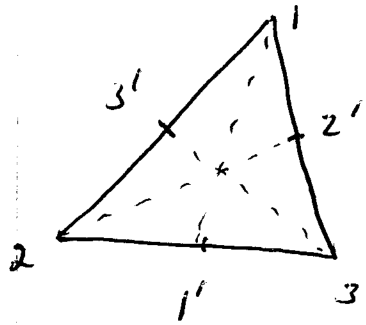
$\{e, a, b, c\}$



	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

Two reflections (13) and (24) and a rotation by π.

Smallest non-abelian group D_3



$(1\,1')$ $(2\,2')$ $(3\,3')$ reflections
rotations by $2\pi/3$, $4\pi/3$

Def: <u>Subgroup</u> subset $H \in G$ which forms a group under the same multiplication law

e.g. $D_2 \ni \{e, a\}$ ← subgroups each is isomorphic to C_2
$\{e, b\}$
$\{e, c\}$

Rearrangement Lemma
if $p, b, c \in G$ and $pb = pc \Rightarrow b = c$

so

if b, c are distinct so are pb and pc.
then multiplication by an element of G against all of G just rearranges the group elements

$\{g_1, \ldots, g_n\} = G \qquad pG = \{g_{p_1}, \ldots, g_{p_n}\}$

$$p = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ p_1 & p_2 & p_3 & \cdots & p_n \end{pmatrix}$$ ← set of permutations
S_n ← symmetric ~~gr~~ or permutation group.

Def: Isomorphism: G isomorphic to G' iff $\exists$ a one-to-one mapping between them that persever the multiplication rule.

$g_i \in G$, $g_i' \in G'$ and $g_1 g_2 = g_3 \Rightarrow g_1' g_2' = g_3'$ and so on

Thm (Cayley)

Every group G of order n is isomorphic to a subgroup of S_n.

Pf: Rearrangement lemma $a \in G \to p_a = \begin{pmatrix} 1 & 2 & \cdots & n \\ a_1 & a_2 & & a_n \end{pmatrix} \in S_n$

where $g_{a_i} = a g_i \quad i = 1, \ldots, n$

say $ab = c$ then

$$p_a p_b = \begin{pmatrix} 1 & 2 & \cdots & n \\ a_1 & a_2 & & a_n \end{pmatrix} \begin{pmatrix} 1 & 2 & \cdots & n \\ b_1 & b_2 & & b_n \end{pmatrix}$$

$$= \begin{pmatrix} b_1 & \cdots & b_n \\ a_1 & & a_n \end{pmatrix} \begin{pmatrix} 1 & \cdots & n \\ b_1 & & b_n \end{pmatrix} = \begin{pmatrix} 1 & \cdots & n \\ a_{b_1} & & a_{b_n} \end{pmatrix}$$

but $g_{a_{b_i}} = a g_{b_i} = ab g_i = c g_i$

so $p_c = \begin{pmatrix} 1 & \cdots & n \\ a_{b_1} & & a_{b_n} \end{pmatrix}$ as well.

<u>Example:</u> $C_3 \; \{e, a, a^2, a^3 = e\}$

<u>cycle notation</u> $p = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 4 & 1 & 2 & 6 \end{pmatrix}$

$1 \to 3 \to 4 \to 1$ $\qquad p = (134)(25)(6)$

same cycle

$2 \to 5 \to 2$

$3 \to 4 \to 1 \to 3$ same cycle

$5 \to 2 \to 5$

$6 \to 6$

$e \cdot C_3 = \to \quad e$ $\qquad (123) \qquad \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$

$a \cdot C_3 \to \{a, a^2, e\}$ $\qquad \cancel{(123)} \; \cancel{(321)}$

$a^2 \cdot C_3 \to \{a^2, e, a\}$ $\qquad (132) \; \cancel{(\;)} \qquad \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$

$\qquad\qquad \overset{\shortparallel}{(321)}$

so $\quad C_3 \cong \{e, (123), (321)$

Classes and Invariant Subgroups

<u>Def:</u> Conjugate elements $a, b \in G$ are conjugate if $\exists$
$p \in G \ni b = pap^{-1} \Longleftrightarrow b \sim a$

Example in S_3 $\quad (12) \sim (31)$ because

$(23)(12)(23)^{-1} = (31)$ $\quad$ check. $\quad \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$

Def: Conjugate class : set of group elements that are cojugate to each other.

e.g. S_3 has three conjugate classes

$\zeta_1 = \{e\}$

$\zeta_2 = \{(12), (23), (13)\}$

$\zeta_3 = \{(123), (321)\}$

why qpq^{-1} just relabels the cycle p_i w q_i.

see

$(23)(12)(23)^{-1} =$

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = (13)$$

so $(12) \sim (13) \Longleftarrow$ These are conjugate.

Def: Conjugate subgroup : If H is a subgroup of G and $a \in G$

Then $H' = \{aha^{-1} ; h \in H\}$ forms a subgroup of G and is a conjugate subgroup.

either $H \cong H'$ (isomorphic) or only have e in common.

Def: Invariant Subgroup $H \subset G$ is an invariant subgroup if it is identical to all its conjugate subgroups.

Example: $C_4 = \{e, a, a^2, a^3, a^4 = e\}$

$H = \{e, a^2\} \subset C_4$ is an invariant subgroup.

a $\quad H' = \{a e a^3,\ a a^2 a^3\} = \{e, a^2\}$

a^2 $\quad H' = \{a^2 e a^2,\ a^2 a^2 a^2\} = \{e, a^2\}$

a^3 $\quad H' = \{a^3 e a,\ a^3 a^2 a\} = \{e, a^2\}$

H is identical to all of its conjugate subgroups.

Every group has at least two invariant subgroups $\{e\}$ and the whole group G.

If non-trivial invariant subgroups exist, the full group can be simplified or factored

<u>Def:</u> Simple and Semi-simple Groups

A group is simple if it has <u>no</u> non-trivial invariant subgroups.

A group is semi-simple if it does not contain any abelian invariant subgroup.

Examples: (a) C_4 contains an abelian invariant subgroup – it is neither simple nor semisimple

(b) C_p with p prime is simple.

2.5 Cosets and Factor (Quotient) Groups

Group

Def: Cosets of $H = \{h_1, \ldots\} \subset G$ and $p \in G$ while $p \notin H$.

$$pH = \{ph_1, \ldots\} \text{ is a } \underline{\text{left coset}} \text{ of } H$$
$$Hp = \{h_1 p, \ldots\} \text{ is a } \underline{\text{right coset}} \text{ of } H.$$

All results for left cosets apply to right ones.

Lemma: Two left cosets of a subgroup H either coincide completely or have no elements in common.

Pf: Let pH and qH be two cosets.

Assume $\exists\, h_i\, h_j \in H \ni ph_i = qh_j$ ← two elements in common.

then $h_i h_j^{-1} = q p^{-1} \iff q p^{-1} \in H \Rightarrow$
$q^{-1}p \in H \Rightarrow$
$q^{-1}p H = H \Rightarrow pH = qH \Rightarrow$ the two cosets are the same.

So Given a subgroup of order n_H, the distinct cosets of H partition the full group G into disjoint sets of n_H each.

Lagrange's Thm: The order of a finite group must be an integer multiple of the order of any of its subgroups.

The cosets (left and right) of an <u>invariant</u> subgroup H are equal.

Recall $pHp^{-1} = H \Rightarrow$

$$pH = Hp. \checkmark$$

Examples w/ S_3

$H_1 \subset S_3$ $\{e, (123), (321)\} \xrightarrow{\text{one left coset}} M = \{(12), (23), (31)\}$

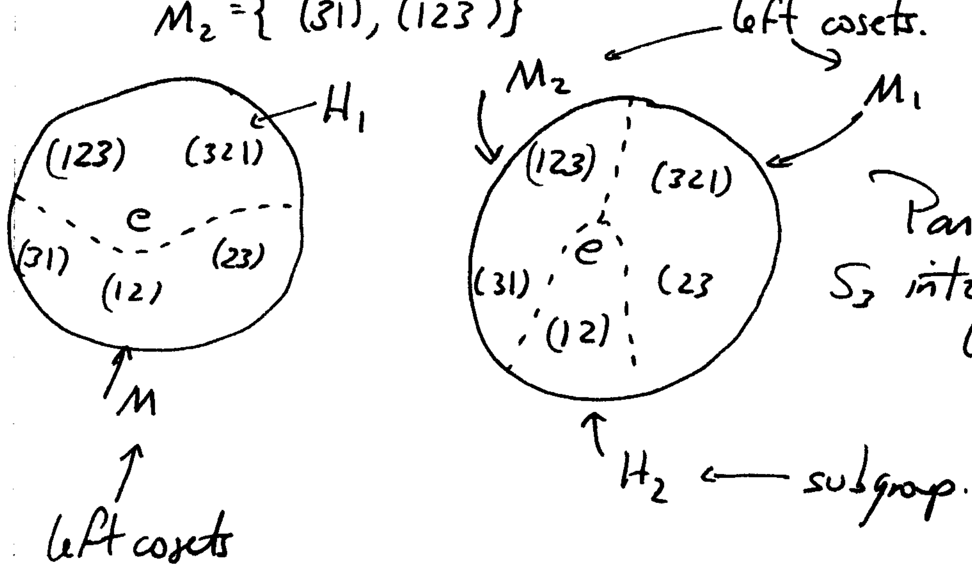
e.g. $(12)(123) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = (23)$ etc.

$H_2 \subset S_3$ $\{e, (12)\}$ has two left cosets

$M_1 = \{(23)\ (321)\}$

$M_2 = \{(31), (123)\}$



Partitioning of S_3 into cosets left.

Consider the cosets of an invariant subgroup as elements of a new group.

$(pH)\cdot(qH) = \text{coset } (pq)H$

we need to show that there is $h_k \in H$ so that

why? $ph_i q h_j = pq h_k$ because

$h_k = (q^{-1}h_i q)h_j$

this is in H since H is an invariant subgroup.

(i) $H = eH$ is the identity element.
(ii) $p^{-1}H = (pH)^{-1}$
(iii) $pH(qH\cdot rH) = (pqr)H$

<u>Thm:</u> If H is an invariant subgroup of G, the set of cosets endowed w/ the multiplication law $pH\cdot qH = pqH$ forms a group. — the factor group or quotient group G/H. It is of order n_G/n_H

Examples $C_4 = \{e, a, a^2, a^3\}$ has an invariant subgroup $H = \{e, a^2\}$ with the cosets

$M = \{a, a^3\}$ and H.

Now $MH = M = HM$ and $MM = H$

So, $C_4/H \cong C_2$

Another example: this time we look at the permutation group or symmetric group. S_3

$H = \{e, (123), (321)\}$ is an invariant subgroup. with two cosets:

H and $M = \{(12), (13), (23)\}$

$HM = H(ij)H = (ij)H = M$

check.

$$(12)(123) = \begin{pmatrix}1&2&3\\2&1&3\end{pmatrix}\begin{pmatrix}1&2&3\\2&3&1\end{pmatrix} = \begin{pmatrix}1&2&3\\1&3&2\end{pmatrix} = (23)$$

So $S_3/H \cong C_2$

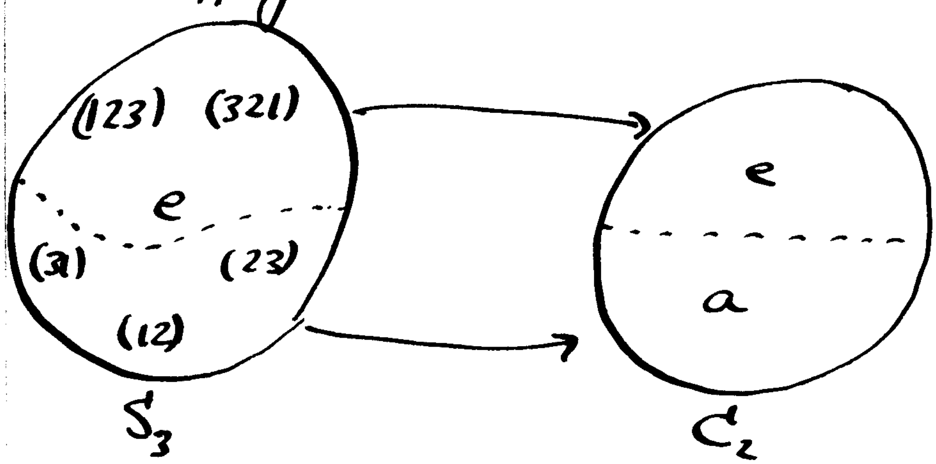
2.6 Homomorphisms

Def: A homomorphism from a group G to another group G' is a mapping which preserves the group multiplication rule.

$g_i \in G$ and $g_j' \in G'$

$$g_3 = g_1 g_2 \longrightarrow g_1' g_2' = g_3'$$

The mapping from S_3 to C_2 is a homomorphism



any three cycle times e or a three cycle is another three cycle. Any two cycle times a two cycle is e. and any two cycle times a three cycle is another two cycle.

$$\begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}\begin{pmatrix}1&2&3\\1&3&2\end{pmatrix} = \begin{pmatrix}1&2&3\\2&1&3\end{pmatrix} = (12)$$

$$(123)\cdot(23) = (12)$$

This reproduces the multiplication table of $C_2=\{e, a, a^2=e\}$.

Thm 2.5 Let f be a homomorphism from G to G'. K = set of all elements mapped to the identity in G'.

$$K=\{a\in G;\ a\xrightarrow{f} e'\in G'\}$$

Then K forms an invariant subgroup of G. And,

$G/K \cong G'$ First show that K is a group

<u>Pf:</u> If $a,b \in K$ then $ab \xrightarrow{f} e' \cdot e' = e' \Rightarrow ab \in K$

Since we have a homomorphism $e \xrightarrow{f} e'$

$g^{-1} \xrightarrow{f} g'^{-1}$ if $g \xrightarrow{f} g'$

Thus $e \in K$ and if $a \in K$ then

$a^{-1} \xrightarrow{f} e'^{-1} = e'$ so K is a subgroup.

Now we need to show that K is an invariant subgroup.

$a \in K$ and $g \in G$ Then

$$gag^{-1} \xrightarrow{f} g'e'g'^{-1} = g'g'^{-1} = e'$$

$\Rightarrow$

$gag^{-1} \in K$ for all $g \in G \Rightarrow K$ is an invariant subgroup.

Now we look at the factor group G/K. Elements of G/K are cosets pK

Consider the mapping $pK \xrightarrow{\rho} p' \in G'$ where $p \xrightarrow{f} p'$

The mapping ρ is one-to-one. To check this we note:

If $\rho(pK) = \rho(qK)$ then $\rho(q^{-1}pK) =$

$\rho(q^{-1}KpK) = \rho(q^{-1}K)\rho(pK)$

Need to think about why this is true.

Why is $\rho(pK)\rho(qK) = \rho(pqK)$?

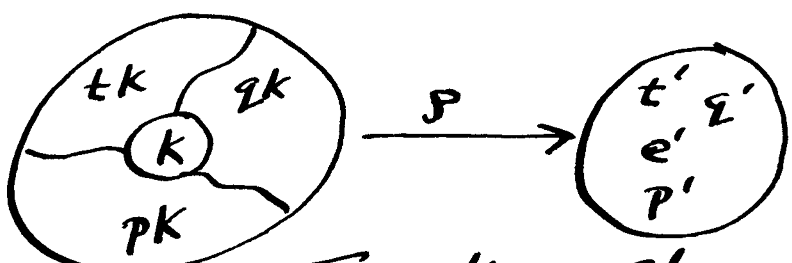
Ans. $pK \xrightarrow{\rho} p'$ if $p \xrightarrow{f} p'$. So any element of the coset pK goes to p' under f $\Rightarrow$ ρ maps ~~p~~ the coset pK to p'.

$\rho(pK)\rho(qK) = p'q'$ and $(pq)' = p'q'$ since f is a homomorphism so we have just proven the point.

means the inverse of $\rho(qK) \in G'$

Then $\rho(q^{-1}pK) = \rho^{-1}(qK)\rho(pK) = e'$ $\Rightarrow$ either $q^{-1}pK = K$ or $qK = pK$ $\therefore$ the mapping is one-to-one.

Since group multiplication is preserved ρ is an isomorphism. $\Rightarrow$ $G/K \cong G'$.



G/K   G'

Isomorphism from G/K to G'

Def: Direct Product Group

Let $H_1, H_2 \subseteq G$ with the properties

(i) elements of H_1 commute w/ elements of H_2

$h_1 h_2 = h_2 h_1 \;\forall\; h_1 \in H,\; h_2 \in H.$

(ii) any element $g \in G$ can be written uniquely as

$g = h_1 h_2$, $h_1 \in H$ $h_2 \in H_2$

Then $G = H_1 \otimes H_2$

example: $C_6 = \{e = a^6, a, a^2, \dots\}$

subgroups $H_1 = \{e, a^3\}$ and $H_2 = \{e, a^2, a^4\}$

$H_1 \cong C_2 \quad H_2 \simeq C_3 \implies$

$C_6 = C_2 \otimes C_3$

If $G = H_1 \otimes H_2$ then both H_1 and H_2 must be invariant subgroups of G.

why? if $g = h_1 h_2 \in G$ $\quad h_1, a_1 \in H_1$

$h_2 \in H_2$

then $g a_1 g^{-1} = \underbrace{h_1 h_2 a_1}_{\text{commute}} h_2^{-1} h_1^{-1} = h_1 a_1 h_1^{-1} \in H_1$ for all g and a_1

so H_1 is an invariant subgroup.

Now we can form the quotient group G/H_1 or G/H_2.

We want to prove that $G/H_1 \simeq H_2$ and $G/H_2 \simeq H_1$

2.15

Consider the mapping $pH_1 \xrightarrow{\rho} h_2$ where $p = h_1 h_2$

$p \in G$

$\Rightarrow p = h_1 h_2$ for some $h_1 \in H_1$, $h_2 \in H_2$

$pH_1 = p h_1 h_2 h_1' = p h_1'' h_2$

Need to show that ρ is a 1-to-1 homomorphism.

1-to-1 say $\rho(pH_1) = \rho(qH_1)$ then:

$p = h_1 h_2$ and $q = h_1' h_2'$ but

$$\rho(pH_1) = \rho(h_2 h_1 H_1) = \rho(h_2 H_1) = h_2$$
$$\rho(qH_1) = \rho(h_2' h_1' H_1) = \rho(h_2' H_1) = h_2'$$

$\Rightarrow h_2 = h_2'$ so

$$pH_1 = h_1 h_2 H_1 = h_2 H_1$$
$$qH_1 = h_1' h_2 H_1 = h_2 H_1$$

$\Rightarrow pH_1 = qH_1$ ✓ one-to-one mapping.

clearly preserves the group multiplication since if $pH_1\, qH_1 = pqH_1$ then $pq = h_2 h_1$

and

$p = h_1^{(p)} h_2^{(p)}$ $q = h_1^{(q)} h_2^{(q)}$ so that

$pq = h_1^{(p)} h_1^{(q)} h_2^{(p)} h_2^{(q)}$ so

$\rho(pqH_1) = h_2^{(p)} h_2^{(q)}$ and $\rho(pH_1)\,\rho(qH_1) = h_2^{(p)} h_2^{(q)}$ ✓

Representations of Groups:

homomorphism U from a group G to linear operators on a vector space $V \Rightarrow U(G)$ forms a representation of the group G of dimension $\dim(V)$.

$$g \in G \xrightarrow{U} U(g) \quad \text{and if } g_1 g_2 = g_3 \text{ then}$$

$$U(g_1)\,U(g_2) = U(g_3)$$

Finite Dimensional Vector space spanned by $\{\hat{e}_1, \ldots, \hat{e}_n\}$ we can define $n \times n$ matrices representing the operators $U(g)$ called $D(g)$ such that

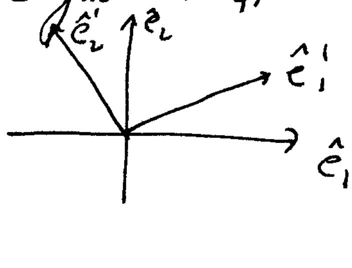
$$U(g)\,|\hat{e}_i\rangle = |\hat{e}_j\rangle D^{j}{}_{i}(g) \qquad \overset{\text{row}}{\downarrow}\; D^{j}{}_{i}(g) \;\nwarrow \text{column}$$

Then $$U(g_1)U(g_2)|\hat{e}_i\rangle = U(g_1)\,|\hat{e}_j\rangle D^{j}{}_{i}(g_2)$$

$$= |\hat{e}_j\rangle D^{j}{}_{k}(g_1)\, D^{k}{}_{i}(g_2)$$

so $$D^{j}{}_{i}(g_1 g_2) = D^{j}{}_{k}(g_1)\, D^{k}{}_{i}(g_2)$$

Example G = group of rotations in the plane about the origin $R(\varphi)$ $\quad 0 \leq \varphi \leq 2\pi$



$$\Rightarrow \hat{e}_1' = \cos\varphi\, \hat{e}_1 + \sin\varphi\, \hat{e}_2$$

$$\hat{e}_2' = -\sin\varphi\, \hat{e}_1 + \cos\varphi\, \hat{e}_2$$

$U(\varphi)|\hat{e}_1\rangle = |\hat{e}_1'\rangle$ etc.

Note $\vec{x}' = D(\varphi)\cdot\vec{x}$ or $x'^j = D(\varphi)^j{}_i x^i$ where we determine D from its action on the unit vectors $|\hat{e}_k\rangle$:

$$x'^j|\hat{e}_j'\rangle = D(\varphi)^j{}_i\, x^i\, |\hat{e}_j\rangle \text{ and}$$

$$\vec{x}'^{\ell} = [U(\varphi)\,\vec{x}]^{\ell} \Rightarrow x^k D^{\ell}{}_k |\hat{e}_\ell\rangle \quad \text{so}$$

$$U(\varphi)|\hat{e}_p\rangle = |\hat{e}_k\rangle D^k{}_p$$

summing on the rows! not on the columns.

<u>Example 2:</u> V_f = complex-valued linear homogeneous functions of real variables (x, y).

$$f(x,y) = ax + by \qquad f \xrightarrow{g\in G} f'(x^1, x^2) \equiv f(x'^1, x'^2)$$

where $\vec{x}' = U(g^{-1})\vec{x}$

Note if
$$f \xrightarrow{g'} f' \qquad f'(\vec{x}) = f[U(g')^{-1}\vec{x}]$$
$$f' \xrightarrow{g''} f'' \qquad f''(\vec{x}) = f'[U(g'')^{-1}\vec{x}]$$

so $f''(\vec{x}) = f[U(g')^{-1}U(g'')^{-1}\vec{x}] = f[U(g''g')^{-1}\vec{x}]$

so ~~$U(g''g') = U(g)$~~ the action of $g''g'$ is the same as the action of g' followed by the action g'' i.e.

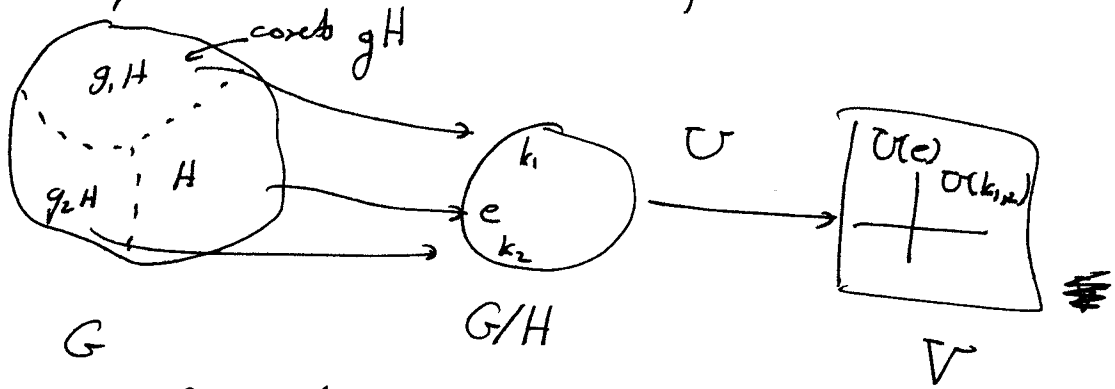
$$f \xrightarrow{g = g''g'} f'' \quad \text{where} \quad f''(\vec{x}) = f[\,U(g)^{-1}\vec{x}\,]$$

<u>Thm 3.1</u> (i) If G has a non-trivial invariant subgroup H then any representation of the factor group $K = G/H$ is also a representation of G. This representation <u>is degenerate</u>

(ii) If $U(G)$ is a degenerate representation of G, then G has at least one invariant subgroup H $\ni$ $U(G)$ defines a faithful representation of G/H.

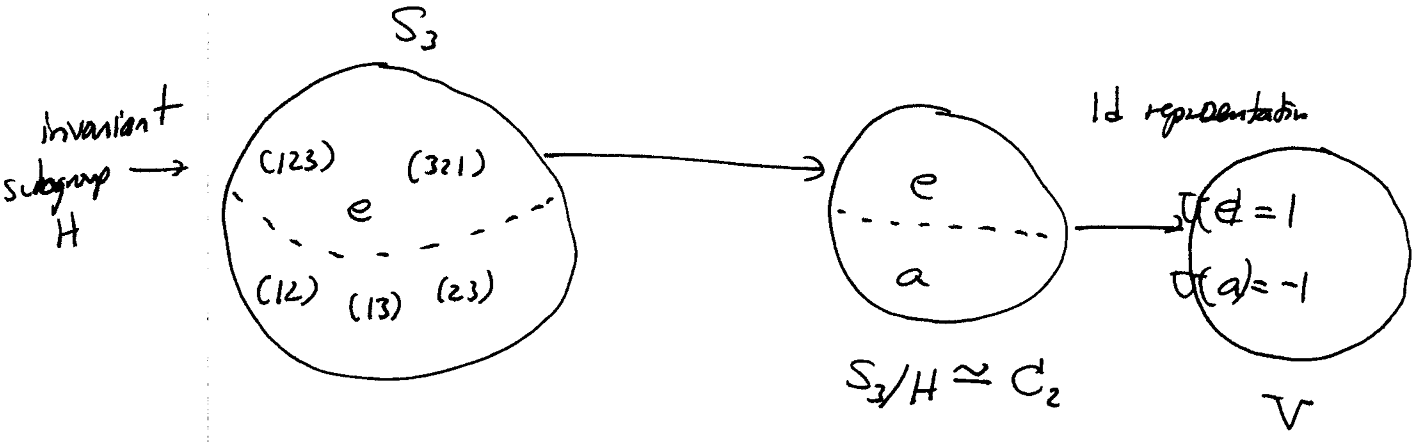
<u>Pf</u>: The mapping $g \in G \longrightarrow k = gH \in K$ followed by $k \to U(k)$ on V is a homomorphism $G \longrightarrow U(k)$

If H is a non-trivial invariant subgroup then $g \longrightarrow k$ is many to one $\Rightarrow$ Not a faithful representation.



2nd part follows from Thm 2.5.

Example:



This induces a non-faithful 1d representation on S_3:

$$\begin{aligned} \{ e \quad (123) \quad (321) \} &\xrightarrow{U'} 1 \\ \{ (12) \quad (13) \quad (23) \} &\longrightarrow -1 \end{aligned}$$

Note that this picks out even and odd (−1) permutations of the numbers 123.

3.2 Irreducible, Inequivalent Representations

<u>Def:</u> Two representations related by a similarity transformation are <u>equivalent</u>

$$U'(G) = S\,U(G)\,S^{-1}$$
$$\Longleftrightarrow U' \text{ and } U \text{ are equivalent.}$$

<u>Def:</u> Character of a representation

$$\chi(g) = \mathrm{Tr}\, U(g)$$

$\Longrightarrow$ All group elements in a conjugate class ~~are equivalent~~. have the same character.

$a \sim g \in G \Rightarrow$
$a = pgp^{-1}$

$\Rightarrow \quad U(a) = U(p)U(g)U(p)^{-1} \Rightarrow$
$\quad Tr\, U(a) = Tr\, U(g).$ ✓

Note also that the trace (or character) is invariant under similarity transformations. So, χ labels conjugate classes only. ← I mean uniquely, in a representation.

Another ~~ε~~ redundancy → <u>Direct Sum Representation</u>

Let $U(G)$ be a representation on V_n and for some choice of basis

$$D(g) = \begin{pmatrix} D_1(g) & 0 \\ 0 & D_2(g) \end{pmatrix} \quad \text{for all } g \in G$$

Then $D(g)$ does not contain any new information not already in $D_1(g)$ and $D_2(g)$.

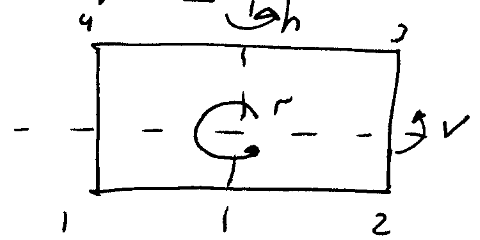
<u>Def:</u> Invariant subspace

subspace V_1 of V which has the property that for all $|x\rangle$ in V_1 $U(g)|x\rangle \in V_1$ for all $g \in G$.

<u>Def:</u> Irreducible Representation:

A rep $U(G)$ on V is irreducible if there is no non-trivial invariant subspace in V wrt $U(G)$

Examples: A Dihedral Group D_2 $\{e, h, v, r\}$



r = rotation by π.

$$D(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad D(h) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad D(v) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$D(r) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

Note that both $\hat{e}_1$ and $\hat{e}_2$ make invariant subspaces of $V = \hat{e}_1, \hat{e}_2$ ← spanned by ...
There is a one-dimensional rep of $\{-1, 1\}$ on each axis that is irreducible.

B) V spanned by $\hat{e}_1$ $\hat{e}_2$ but the group $R(2)$
the vectors
$$\hat{e}_\pm = \frac{1}{\sqrt{2}} (\mp \hat{e}_1 + i \hat{e}_2)$$ form each a 1-d invariant subspace
$$U(\varphi) \hat{e}_\pm = \hat{e}_\pm e^{\mp i \varphi}$$
in the new basis
$$D'(\varphi) = \begin{pmatrix} e^{-i\varphi} & 0 \\ 0 & e^{i\varphi} \end{pmatrix}$$

Unitary Representations:

Thm: If a unitary representation is reducible, then it is also decomposable (i.e. fully reducible)

Pf: Let $U(G)$ be a reducible rep of G on the inner product space V

V_1 is an invariant subspace $[\dim(V_1) = n_1]$ wrt $U(G)$

Take an orthonormal basis $\{\hat{e}_i\}$ $i = 1, \dots, n$ for V

$\ni$ $i = 1, \dots, n_1$ spans V_1 and $i = n_1+1, \dots, n$ spans V_2, the orthogonal complement.

We need to show that V_2 is also invariant wrt $U(G)$

$$|\hat{e}_i(g)\rangle = U(g)|\hat{e}_i\rangle \in V_1 \quad \text{for } i = 1, \dots, n_1$$

Since $U(G)$ is unitary $\langle \hat{e}_j(g) | \hat{e}_i(g) \rangle = 0$ for all $j = n_1+1, \dots, n$ and all $i = 1, \dots, n_1$

Thus any vector $\vec{x}$ in V_2 remains there under the action $U(G)$. $\boxed{\text{We say } \underbrace{V}_{D(G)} = V_1 \oplus V_2 .}$

Thm 3.3 Every representation of a finite group on an inner product space is equivalent to a unitary rep.

Need to find S so that $S D(G) S^{-1} = \underbrace{U(G)}_{\text{unitary}}$

for all $g \in G$.

S can be chosen to satisfy

$$(x, y) \equiv \langle S x | S y \rangle = \sum_g \langle D(g) x | D(g) y \rangle$$

~~Note~~ Note: 1 (x,y) satisfies the axioms of a scalar product.

$$(x,y) = (y,x)^*$$
$$(x,x) \geqslant 0 \text{ and } = 0 \text{ iff } x = 0.$$

2 S is a transformation from a basis orthonormal wrt $\langle | \rangle$ to one orthonormal wrt $(,)$.

Shur's Lemmas

1. Let $U(G)$ be an irreducible rep of G on V and let A be an operator on V

if $[A, U(g)] = 0 \quad \forall g \in G$ then

$A = \lambda E$, where λ = a number

E = Identity operator.

$\Rightarrow$ Irreducible reps of an abelian group have dimension 1.

Why? Because $U(g')$ commutes w/ all $U(g)$, $g \in G$

$\Rightarrow \quad U(g') = \lambda_{g'} E.$

Pf of Shur's First Lemma:

Take $U(G)$ to be unitary and make A Hermitian if A is not can work ~~w~~ with

$$A_+ = \frac{A + A^+}{2} \quad A_- = \frac{A - A^+}{2i}$$

Take the basis of V to be eigenvectors of A.

to label vectors in degenerate subspaces of A.

(3.9)

$|u_{\alpha,i}\rangle$ ← may be chosen to be orthonormal.

$$A|u_{\alpha,i}\rangle = |u_{\alpha,i}\rangle \lambda_i$$

Note: for any i $\{\hat{u}_{\alpha,i};\ \alpha = 1,2\ldots\} = V^i$ are invariant subspaces wrt $U(G)$.

$$A\, U(g)|u_{\alpha,i}\rangle = U(g) A |u_{\alpha,i}\rangle = \lambda_i U(g)|u_{\alpha,i}\rangle$$

so $U(g)|u_{\alpha,i}\rangle \in V^i$ ✓

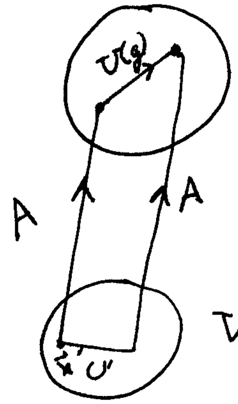
But $U(G)$ is irreducible on $V \Rightarrow V^i$ must be V
$\Rightarrow A$ has only one eigenvalue $\Rightarrow A = \lambda E$.

Shur's 2nd Lemma:

Let $U(G)$ and $U'(G)$ be two irreducible reps of G on V and V' respectively.
Let A be linear transformation from V' to V such that
$$AU'(g) = U(g)A \text{ for all } g \in G.$$

Then either $A = 0$ or
V is isomorphic to V and $U(G)$ is equivalent to $U'(G)$.

3.10



V

R = the range of A

$R = \{\vec{x} \in V \mid \vec{x} = A\vec{x}';\ \vec{x}' \in V'\}$

V' (i) R is an invariant subspace of V wrt $U(G)$

$\Longleftrightarrow\ |x\rangle \in V \qquad U(g)|x\rangle = U(g)A|x'\rangle = AU'(g)|x'\rangle$

$= A|U'(g)x'\rangle \in R$ for all $g \in G$.

If $U(G)$ is an irreducible representation either $R=0$ or $R=V \Rightarrow$ In the first case $A=0$. ~~In the second case~~.

(ii) Let $N' \subset V'$ be the null space of A.

$N' = \{|x'\rangle \in V' \mid A|x'\rangle = 0\}$

N' is an invariant subspace of V' wrt $U'(G)$

if $|x'\rangle \in N'$ then $AU'(G)|x'\rangle = U(g)A|x'\rangle = U(g)|0\rangle = 0$

Since $U'(G)$ is irreducible either $N' = V'$ (hence $A=0$) or $N' = 0$. If the later

$A|x'\rangle = A|y'\rangle \Rightarrow |x'\rangle = |y'\rangle$ i.e. A is a 1-to-1 mapping.

Thus A is an isomorphism from V' to V^* where $U(G) = AU'(G)A^{-1}$ or $A=0$.

Orthonormality and Completeness Relations of Irreducible Representations of Matrices.

n_G = order of group G.

$\overset{\mu}{D}(g)$ = matrix corresponding to $g \in G$ in the μ-representation wrt an orthonormal basis

n_μ = dimension of the μ-representation.

n_i = # elements in class ξ_i w/ character χ_i^μ

n_c = # classes in G

Thm 3.5 Orthonormality of Irreducible Reps.

$$\frac{n_\mu}{n_G} \sum_g D_\mu^{+}(g)^k{}_i \, \overset{\nu}{D}(g)^j{}_l = \delta_\mu^\nu \, \delta_i^j \, \delta_l^k$$

applies to irreducible representations

for ν, l, j $\quad \overset{\nu}{D}(g)^j{}_l \sqrt{\frac{n_\nu}{n_G}}$ can be thought of as a n_G component vector

Examples

1. C_2 $\quad d_1$ identity rep. $\quad (e,a) \xrightarrow{d_1} (1,1)$

another irreducible rep must be orthogonal to this one so
$d_2 \quad (e,a) \xrightarrow{d_2} (1,-1)$ works.

$\mu \backslash g$	e	a
1	1	1
2	1	-1

2. Dihedral Group D_2

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

trivial rep: $(e, a, b, c) \xrightarrow{d_1} (1,1,1,1)$

The elements $\{e,a\}$ form an invariant subgroup $\cong C_2$
what are the cosets?
① $b\{e,a\} = \{b,c\}$
$c\{e,a\} = \{c,b\}$
② $e\{e,a\} = \{e,a\}$
$a\{e,a\} =$

so the factor group $D_2/C_2 = \{(e,a), (b,c)\} \cong C_2$

This factor group has two inequivalent representations. From Thm 3.1 there are two induced representations of the full group D_2

$$\{e,a,b,c\} \longrightarrow \{1,1,1,1\}$$
$$\{e,a,b,c\} \rightarrow \{1,1,-1,-1\}$$

There are two more invariant subgroups $\{e,b\}$ and $\{e,c\}$

$\Longrightarrow$ Doing the same thing generates two more irreducible representations

$\{e, a, b, c\} \xrightarrow{d_3} \{1, -1, 1, -1\}$; $\{e, a, b, c\} \xrightarrow{d_4} \{1, -1, -1, 1\}$

So we have the following table of inequivalent Irreducible representations of D_2:

$\mu \backslash g$	e	a	b	c
1	1	1	1	1
2	1	1	-1	-1
3	1	-1	1	-1
4	1	-1	-1	1

Proof of Thm 3.5

X is any $n_\mu \times n_\nu$ matrix.

Define $$M_X = \sum_g D_\mu^+(g) \, X \, D^\nu(g) \quad ; \quad \text{Recall } D^+ = D^{-1}$$

Then $D^+_\mu(p) M_X D^\nu(p) = M_X$ by the rearrangement lemma.
for any $p \in G$

By Schur's 2nd lemma $M_X = 0$ and $\mu \neq \nu$ or
$\mu = \nu$ and $M_X = c_X \, E$

const Identity

Now choose X to be a $n_\mu \times n_\nu$ matrix with

$$(X^k_\ell)^i{}_j = \delta^k_j \delta^i_\ell$$

runs over representations matrix indices

Then:

$$(M^k_\ell)^m{}_n = \sum_g D^\dagger_\mu(g)^m{}_i \,(X^k_\ell)^i{}_j\, D^\nu(g)^j{}_n$$

$$= \sum_g D^\dagger_\mu(g)^m{}_\ell\, D^\nu(g)^k{}_n$$

needs to be the same representation or the LHS is zero.

for $\mu = \nu$ the LHS $= c^k_\ell\, \delta^m_n$

constants.

Taking the trace of both sides:

LHS $= n_\mu c_\ell$

RHS $= \sum_g [D^\mu(g) D^\dagger_\mu(g)]^k{}_\ell = \delta^k_\ell\, n_G$

one term for each group element.

$\Rightarrow \quad c^k_\ell = n_G/n_\mu\, \delta^k{}_\ell \leftarrow$ Book has n_μ / n_G ??

If one introduces the notation $\langle g | \mu, j, \ell \rangle = \sqrt{\frac{n_\mu}{n_G}}\, D^\mu(g)^j{}_\ell$

Then this then takes the form:

$$\boxed{\sum_g \langle \mu, i, k | g \rangle \langle g | \nu, j, \ell \rangle = \delta^\nu_\mu\, \delta^j_i\, \delta^k_\ell}$$

Completeness of Irreducible Representation Matrices.

(i) $$\sum_{\mu} n_{\mu}^{2} = n_G$$

(ii) $$\sum_{\mu,\ell k} \frac{n_\mu}{n_G} D^{\mu}(g)^{\ell}{}_{k}\, D^{\dagger}_{\mu}(g')^{k}{}_{\ell} = \delta_{gg'}$$

or

$$\sum_{\mu\,\ell,k} \langle g|\mu,\ell,k\rangle\langle \mu,\ell,k|g'\rangle = \delta_{gg'}$$

using $\langle g|\mu,j,\ell\rangle = \sqrt{\frac{n_\mu}{n_G}}\, D^{\mu}(g)^{j}{}_{\ell}$